Math 32A, Lecture 1
Multivariable Calculus

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: Solutions

UID: __________

Section: __________

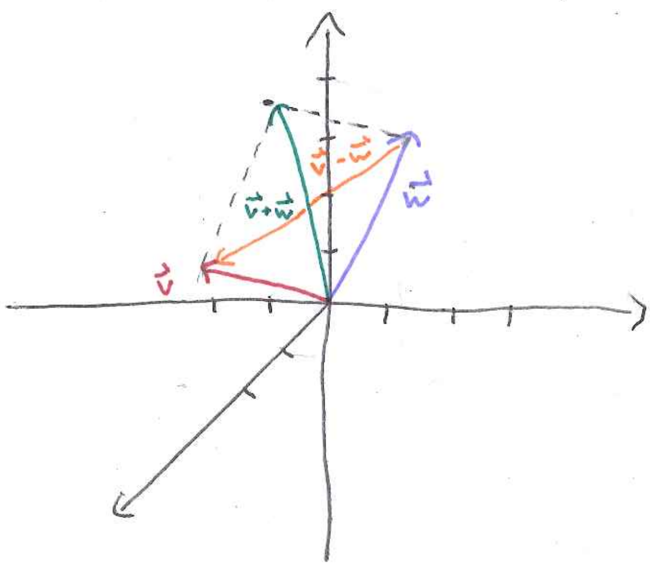
Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Draw the vectors $\mathbf{v} = \langle 2, -1, 2\rangle$ and $\mathbf{w} = \langle 1, 2, 4\rangle$. Sketch $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ on your picture.

(b) [5pts.] What is the area of the parallelogram spanned by the unit vectors $\mathbf{e_v}$ and $\mathbf{e_w}$ in the direction of $\mathbf{v}$ and $\mathbf{w}$? [Hint: There is a fast way to do this, using the fact that $\sin^2(\theta) = 1 - \cos^2(\theta)$.]

(a)



(b) Since $\vec{e}_{\vec{v}}$ and $\vec{e}_{\vec{w}}$ are unit vectors, $\|\vec{e}_{\vec{v}} \times \vec{e}_{\vec{w}}\| = \|\vec{e}_{\vec{v}}\| \|\vec{e}_{\vec{w}}\| \sin\theta$

$= \sin\theta$

where θ is the angle between $\vec{e}_{\vec{v}}$ and $\vec{e}_{\vec{w}}$, or equivalently between $\vec{v}$ and $\vec{w}$. Now $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \dfrac{2-2+8}{\sqrt{9}\sqrt{21}} = \dfrac{8}{\sqrt{189}}$

So $\sin^2\theta = 1 - \dfrac{64}{189} = \dfrac{125}{189} \Rightarrow \sin\theta = \sqrt{\dfrac{125}{189}} = \|\vec{e}_{\vec{v}} \times \vec{e}_{\vec{w}}\|$ is the area of the parallelogram spanned by $\vec{e}_{\vec{v}}$ and $\vec{e}_{\vec{w}}$.

Problem 2.

(a) [5pts.] Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$. Suppose that $\mathbf{u} = \mathbf{u}_{\|\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}$, where $\mathbf{u}_{\|\mathbf{v}}$ is the projection of $\mathbf{u}$ to $\mathbf{v}$ and $\mathbf{u}_{\perp\mathbf{v}} = \mathbf{u} - \mathbf{u}_{\|\mathbf{v}}$ is orthogonal to $\mathbf{v}$. Use the properties of the cross product to prove that $\mathbf{v} \times \mathbf{u} = \mathbf{v} \times \mathbf{u}_{\perp\mathbf{v}}$.

(b) [5pts.] Find the projection of $\mathbf{u} = \langle 1, 3, 2\rangle$ to $\mathbf{v} = \langle 0, 9, 6\rangle$, and use part (a) to quickly compute $\mathbf{u} \times \mathbf{v}$.

(a) Observe that $\vec{v} \times \vec{u} = \vec{v} \times (\vec{u}_{\|\vec{v}} + \vec{u}_{\perp\vec{v}})$

$$= (\vec{v} \times \vec{u}_{\|\vec{v}}) + (\vec{v} \times \vec{u}_{\perp\vec{v}})$$

Now $\vec{u}_{\|\vec{v}}$ is a scalar multiple of $\vec{v}$ $\Rightarrow$ $\vec{v} \times \vec{u}_{\|\vec{v}} = 0$.

So $\vec{v} \times \vec{u} = \vec{v} \times \vec{u}_{\perp\vec{v}}$.

(b) $$\vec{u}_{\|\vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{0+27+12}{0+81+36} \vec{v} = \frac{39}{117} \vec{v} = \frac{1}{3} \vec{v} = \langle 0, 3, 2 \rangle$$

$$\vec{u}_{\perp\vec{v}} = \langle 1, 0, 0 \rangle$$

$$\vec{v} \times \vec{u} = \vec{v} \times \vec{u}_{\perp\vec{v}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 9 & 6 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 9 & 6 \\ 0 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 6 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 9 \\ 1 & 0 \end{vmatrix} \vec{k}$$

$$= \langle 0, 6, -9 \rangle$$

Problem 3.

Let $P = (0, 1, 4)$, $Q = (2, 3, 1)$, and $R = (3, -1, -2)$.

(a) [5pts.] Find an equation for the plane containing P, Q, and R.

(b) [5pts.] Give a parametrization of the intersection of the plane from part (a) with the cylinder $(x-1)^2 + (y-4)^2 = 25$.

(a) $\vec{PQ} = \langle 2, 2, -3 \rangle$

$\vec{PR} = \langle 3, -2, -6 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -3 \\ 3 & -2 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 \\ -2 & -6 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -3 \\ 3 & -6 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} \vec{k}$$

$$= (-12-6)\vec{i} - (-12+9)\vec{j} + (-4-6)\vec{k}$$

$$= -18\vec{i} + 3\vec{j} - 10\vec{k}$$

Our plane is $-18x + 3y - 10z = d$ and contains $(0, 1, 4)$

$\Rightarrow d = -18(0) + 3(1) - 10(4) = -37$

$$\boxed{-18x + 3y - 10z = -37}$$

(b) The cylinder contains all points (x, y, z) with

$\begin{cases} x = 5\cos t + 1 \\ y = 5\sin t + 4 \end{cases}$, So on its intersection with the plane,

we have $-18(5\cos t + 1) + 3(5\sin t + 4) - 10z = -37$

continued

$$\Rightarrow -90\cos t - 18 + 15\sin t + 12 - 10z = -37$$

$$\Rightarrow z = \frac{-1}{10}(90\cos t - 15\sin t - 31)$$

$$\begin{cases} x(t) = 5\cos t + 1 \\ y(t) = 5\sin t + 4 \\ z(t) = \frac{-1}{10}(90\cos t - 15\sin t - 31) \end{cases}$$

Problem 4.

Consider the following four vector equations for lines.

$$\begin{aligned} \mathbf{r}_1(t) &= \langle 1,2,1\rangle + t\langle -3,3,6\rangle \\ \mathbf{r}_2(s) &= \langle 7,4,3\rangle + s\langle 2,-2,-4\rangle \\ \mathbf{r}_2(u) &= \langle 1,2,1\rangle + u\langle 4,3,2\rangle \\ \mathbf{r}_4(v) &= \langle 4,-1,-5\rangle + v\langle .5,-.5,-1\rangle \end{aligned}$$

(a) [5pts.] Determine which two equations above parametrize the same line.

(b) [5pts.] Find the point of intersection between the line parametrized by $\mathbf{r}_1(t)$ and the line parametrized by $\mathbf{r}_5(w) = \langle 4,5,1\rangle + w\langle 1,0,-1\rangle$, and find the angle between the ~~vectors~~ lines at the point of intersection.

(a) We see that $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_4$ have the same direction vector up to scaling. Hence any two of that that share a point are the same line.

Now, $\langle 4,-1,-5\rangle$ is on $\vec{r}_4$. Asking if it is on $\vec{r}_1$ is the same as asking if there is a t such that

$$\langle 4,-1,-5\rangle = \langle 1,2,1\rangle + t\langle -3,3,6\rangle$$

$$\Rightarrow \langle 3,-3,-6\rangle = t\langle -3,3,6\rangle$$

We see $t=-1$ is a solution. So $\vec{r}_1$ and $\vec{r}_4$ parametrize the same line.

(b) We solve

$$\langle 1,2,1\rangle + t\langle -3,3,6\rangle = \langle 4,5,1\rangle + w\langle 1,0,-1\rangle$$

$$\Rightarrow t\langle -3,3,6\rangle = \langle 3,3,0\rangle + w\langle 1,0,-1\rangle$$

$$\Rightarrow \begin{cases} -3t = 3+w \;\} \text{ Satisfied by } t=1,\ w=-6 \\ 3t = 3 \rightsquigarrow t=1 \\ 6t = -w \rightsquigarrow w=-6 \end{cases}$$

$$\boxed{\langle -2,5,7\rangle}$$

ctd

The angle between the lines is the angle between their direction vectors $\langle -3, 3, 6 \rangle$ and $\langle 1, 0, -1 \rangle$.

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_5}{\|\vec{v}_1\| \|\vec{v}_5\|} = \frac{-3+0-6}{\sqrt{54}\sqrt{2}} = \frac{-9}{\sqrt{108}} = \frac{-9}{\sqrt{9 \cdot 12}} = \frac{-9}{6\sqrt{3}} = \frac{-\sqrt{3}}{2}$$

$$\boxed{\theta = \frac{5\pi}{6}}$$

Problem 5.

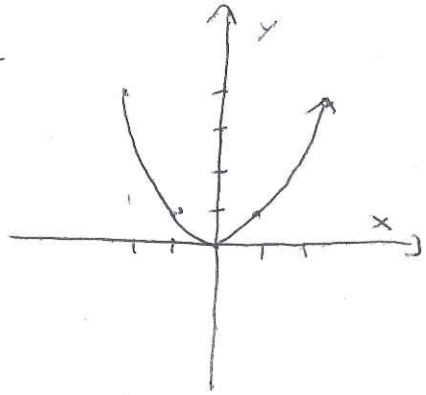
Consider the vector-valued function $\mathbf{r}(t) = \langle t, t^2, \sin(\pi t)\rangle$.

(a) [5pts.] Draw the projections of $\mathbf{r}(t)$ to the three coordinate planes, and use these to give a sketch of the space curve determined by $\mathbf{r}(t)$.

(b) [5pts.] Find the equation of the tangent line to $\mathbf{r}(t)$ at $t = \frac{1}{2}$.

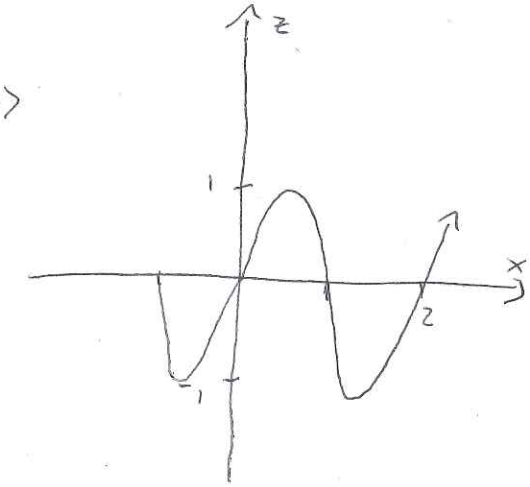
(a) xy-plane

$\langle t, t^2 \rangle$



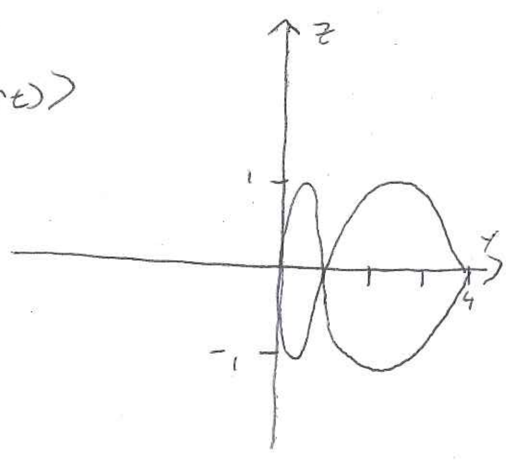
xz-plane

$\langle t, \sin(\pi t)\rangle$

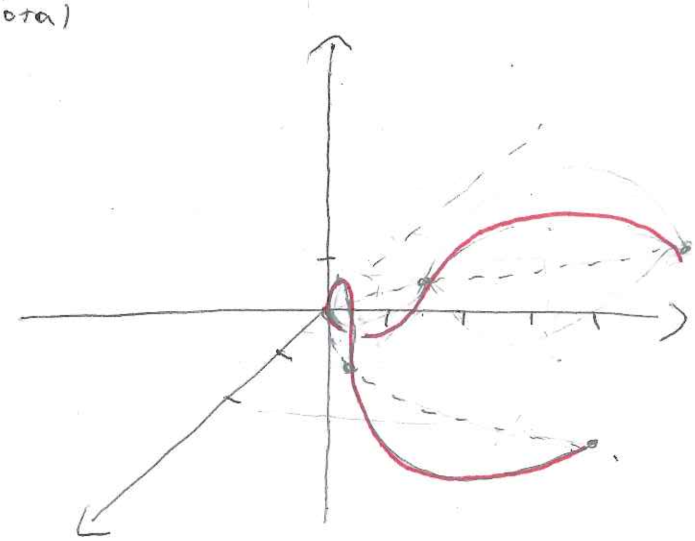


yz-plane

$\langle t^2, \sin(\pi t)\rangle$



Total



(b) $\vec{r}\,'(t) = \langle 1, 2t, \pi\cos(\pi t)\rangle$

$\vec{r}(.5) = \langle .5, .25, 1\rangle \qquad \vec{r}\,'(.5) = \langle 1, 1, 0\rangle$

Tangent line is $\boxed{\vec{s}(t) = \langle .5, .25, 1\rangle + t\langle 1, 1, 0\rangle}$